# Multivariable link invariants and renormalized quantum dimension

Cristina Ana Maria Anghel

Paris Diderot University

ECSTATIC
Imperial College London
June 11-12, 2015

# Abstract

- We intend to describe a family of multivariable link invariants introduced by N. Geer and B. Patureau.
- The algebraic input will be a category of representations associated to a super Lie algebra of type one.
- The key point is to define a "renormalized quantum dimension" of a module and use it instead of the usual quantum dimension in a Reshetikhin-Turaev type construction.
- We will explain this idea and the definition of the multivariable link invariants.

# Abstract

- We intend to describe a family of multivariable link invariants introduced by N. Geer and B. Patureau.
- The algebraic input will be a category of representations associated to a super Lie algebra of type one.
- The key point is to define a "renormalized quantum dimension" of a module and use it instead of the usual quantum dimension in a Reshetikhin-Turaev type construction.
- We will explain this idea and the definition of the multivariable link invariants.

# Abstract

- We intend to describe a family of multivariable link invariants introduced by N. Geer and B. Patureau.
- The algebraic input will be a category of representations associated to a super Lie algebra of type one.
- The key point is to define a "renormalized quantum dimension" of a module and use it instead of the usual quantum dimension in a Reshetikhin-Turaev type construction.
- We will explain this idea and the definition of the multivariable link invariants.

# Abstract

- We intend to describe a family of multivariable link invariants introduced by N. Geer and B. Patureau.
- The algebraic input will be a category of representations associated to a super Lie algebra of type one.
- The key point is to define a "renormalized quantum dimension" of a module and use it instead of the usual quantum dimension in a Reshetikhin-Turaev type construction.
- We will explain this idea and the definition of the multivariable link invariants.

# Outline

1. Renormalized Reshetikhin-Turaev type construction
   - Motivation
   - Classical Reshetikhin-Turaev invariants
   - Renormalized construction
2. Multivariable invariants
   - Geer and Patureau's Multivariable Invariants
   - Relations with other known invariants
   - Further directions

# Motivation

- In 1991, Reshetikhin and Turaev defined one construction which starts with any Ribbon category and and gives colored link invariants.
- They use in the definition the notion of quantum dimension of a module.
- Usually, people apply this construction for categories which come from the representation theory of some Hopf algebras(quantum groups).
- If we start with $g$ a super-Lie algebra of type one, and we look at the quantum enveloping algebra, this is a quantum group, but we have some issues.

# Motivation

- In 1991, Reshetikhin and Turaev defined one construction which starts with any Ribbon category and and gives colored link invariants.
- They use in the definition the notion of quantum dimension of a module.
- Usually, people apply this construction for categories which come from the representation theory of some Hopf algebras(quantum groups).
- If we start with $g$ a super-Lie algebra of type one, and we look at the quantum enveloping algebra, this is a quantum group, but we have some issues.

# Motivation

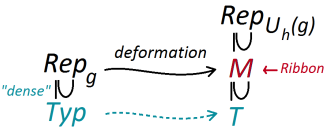
- In 1991, Reshetikhin and Turaev defined one construction which starts with any Ribbon category and and gives colored link invariants.
- They use in the definition the notion of quantum dimension of a module.
- Usually, people apply this construction for categories which come from the representation theory of some Hopf algebras(quantum groups).
- If we start with $g$ a super-Lie algebra of type one, and we look at the quantum enveloping algebra, this is a quantum group, but we have some issues.

# Motivation

- In 1991, Reshetikhin and Turaev defined one construction which starts with any Ribbon category and and gives colored link invariants.
- They use in the definition the notion of quantum dimension of a module.
- Usually, people apply this construction for categories which come from the representation theory of some Hopf algebras(quantum groups).
- If we start with $g$ a super-Lie algebra of type one, and we look at the quantum enveloping algebra, this is a quantum group, but we have some issues.

# Motivation II

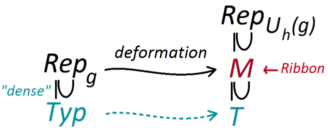
- We have a method to produce a Ribbon category using its representation theory.




- However, if we look at the Reshetikhin-Turaev construction for M, this leads to invariants for M-colored links that vanish on any link which has at least one strand colored with a T-color.
- Idea: Geer and Patureau modified this construction, using a "renormalized quantum dimension" in order to obtain non-vanishing invariants.

# Motivation II

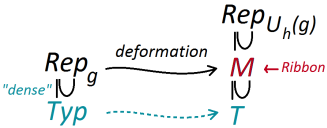
- We have a method to produce a Ribbon category using its representation theory.




- However, if we look at the Reshetikhin-Turaev construction for M, this leads to invariants for M-colored links that vanish on any link which has at least one strand colored with a T-color.
- Idea: Geer and Patureau modified this construction, using a "renormalized quantum dimension" in order to obtain non-vanishing invariants.

# Motivation II

- We have a method to produce a Ribbon category using its representation theory.




- However, if we look at the Reshetikhin-Turaev construction for M, this leads to invariants for M-colored links that vanish on any link which has at least one strand colored with a T-color.
- Idea: Geer and Patureau modified this construction, using a "renormalized quantum dimension" in order to obtain non-vanishing invariants.

## Definition

- Let $\mathcal{C}$ a strict monoidal category.
- A braiding $C$ is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:
  $C_{U,V\otimes W} = (Id_V \otimes C_{U,W}) \circ (C_{U,V} \otimes Id_W)$
  $C_{U\otimes V,W} = (C_{U,W} \otimes Id_V) \circ (Id_U \otimes C_{V,w})$.
- If $\mathcal{C}$ has the brading $C$, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$: $\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.
- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*, \quad d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties: $(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$
  $(d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}$.
- The duality is said to be compatible with the brading and the twist if: $\forall V \in \mathcal{C}$, $(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V$. A category with a brading, a twist and a compatible duality is called a Ribbon Category.

## Definition

- Let $\mathcal{C}$ a strict monoidal category.
- A braiding $C$ is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:
  $C_{U,V\otimes W} = (Id_V \otimes C_{U,W}) \circ (C_{U,V} \otimes Id_W)$
  $C_{U\otimes V,W} = (C_{U,W} \otimes Id_V) \circ (Id_U \otimes C_{V,w})$.
- If $\mathcal{C}$ has the brading $C$, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$: $\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.
- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$, $d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties: $(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$
  $(d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}$.
- The duality is said to be compatible with the brading and the twist if: $\forall V \in \mathcal{C}$, $(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V$. A category with a brading, a twist and a compatible duality is called a Ribbon Category.

## Definition

- Let $\mathcal{C}$ a strict monoidal category.
- A braiding $C$ is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:
  $C_{U,V\otimes W} = (Id_V \otimes C_{U,W}) \circ (C_{U,V} \otimes Id_W)$
  $C_{U\otimes V,W} = (C_{U,W} \otimes Id_V) \circ (Id_U \otimes C_{V,w})$.
- If $\mathcal{C}$ has the brading $C$, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$: $\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.
- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$, $d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties: $(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$
  $(d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}$.
- The duality is said to be compatible with the brading and the twist if: $\forall V \in \mathcal{C}$, $(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V$. A category with a brading, a twist and a compatible duality is called a Ribbon Category.

## Definition

- Let $\mathcal{C}$ a strict monoidal category.
- A braiding $C$ is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:
  $C_{U,V\otimes W} = (Id_V \otimes C_{U,W}) \circ (C_{U,V} \otimes Id_W)$
  $C_{U\otimes V,W} = (C_{U,W} \otimes Id_V) \circ (Id_U \otimes C_{V,w})$.
- If $\mathcal{C}$ has the brading $C$, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$: $\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.
- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*,\ \ d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties: $(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$
  $(d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}$.
- The duality is said to be compatible with the brading and the twist if: $\forall V \in \mathcal{C}$, $(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V$. A category with a brading, a twist and a compatible duality is called a Ribbon Category.

## Definition

- Let $\mathcal{C}$ a strict monoidal category.
- A braiding $C$ is a natural set of isomorphisms $C = \{C_{V,W} \mid C_{V,W} : V \otimes W \to W \otimes V, V, W \in \mathcal{C}\}$ such that for any $U, V, W \in \mathcal{C}$ the following relations hold:
$C_{U,V\otimes W} = (Id_V \otimes C_{U,W}) \circ (C_{U,V} \otimes Id_W)$
$C_{U\otimes V,W} = (C_{U,W} \otimes Id_V) \circ (Id_U \otimes C_{V,w})$.
- If $\mathcal{C}$ has the brading $C$, a twist means a family of natural isomorphisms $\Theta = \{\theta_V \mid \theta_V : V \to V, V \in \mathcal{C}\}$ such that $\forall V, W \in \mathcal{C}$: $\theta_{V\otimes W} = C_{W,V} \circ C_{V,W}(\theta_V \otimes \theta_W)$.
- We have a duality in $\mathcal{C}$ if for any $V \in \mathcal{C}$ there is $V^* \in \mathcal{C}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$, $d'_V : V \otimes V^* \to \mathbf{1}$ with the following properties: $(Id_V \otimes d_V) \circ (b_V \otimes Id_V) = Id_v$
$(d_V \otimes Id_{V^*}) \circ (Id_{V^*} \otimes b_V) = Id_{V^*}$.
- The duality is said to be compatible with the brading and the twist if: $\forall V \in \mathcal{C}$, $(\theta_V \otimes Id_{V^*})b_V = (Id_V \otimes \theta_{V^*})b_V$. A category with a brading, a twist and a compatible duality is called a Ribbon Category.

# Category of framed colored tangles

**Definition**

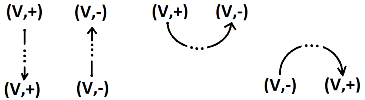
Consider $\mathcal{C}$ a category. The category of $\mathcal{C}$-colored framed tangles $\mathcal{T}_{\mathcal{C}}$ is defined as follows:

$Ob(\mathcal{T}_{\mathcal{C}}) = \{(V_1, \epsilon_1), ..., (V_m, \epsilon_m) \mid m \in \mathbb{N}, \epsilon_i \in \{\pm 1\}, V_i \in \mathcal{C}\}$.

$Morph(\mathcal{T}_{\mathcal{C}})((V_1, \epsilon_1), ..., (V_m, \epsilon_m), (W_1, \delta_1), ..., (W_n, \delta_n)) =$

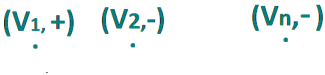
$$\frac{\mathcal{C}\,colored\ framed\ tangles\ T : (V_1, \epsilon_1), ..., (V_m, \epsilon_m) \to (W_1, \delta_1), ..., (W_n, \delta_n)}{isotopy}.$$

- *Observation* : The tangles have to respect the colors $V_i$. Once we have such a tangle, it has an induced orientation, coming from the signs $\epsilon_i$, using the following conventions:

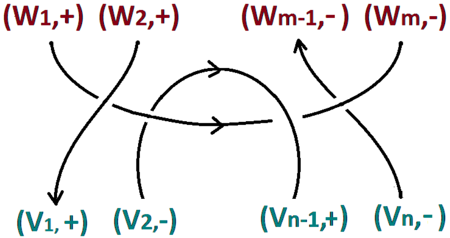


# Example

$V \in Ob$




$T \in Morph(V,W)$



# Reshetikhin-Turaev functor

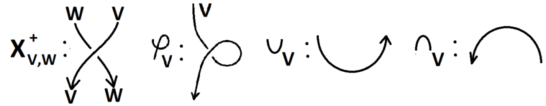
- Aim: Starting with any Ribbon Category $\mathcal{C}$, we'll define a functor from the category of framed $\mathcal{C}$-colored tangles to $\mathcal{C}$.

**Theorem (Reshetikhin-Turaev)**

Consider( $\mathcal{C}$, $C$, $\Theta$, $b$, $d'$ ) a Ribbon category. Then there exist an unique functor $F_{\mathcal{C}} : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}$ which is monoidal and satisfies the following local relations for any $V, W \in \mathcal{C}$:

1) $F((V,+)) = V \quad F((V,-)) = (V)^*$

2) $F(X^+_{V,W}) = C_{V,W} \quad F(\varphi_V) = \theta_V \quad F(\cup_V) = b_V \quad F(\cap_V) = d'_V$, where



# Super Lie algebras of type I

**Definition**

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,\ ] : g^{\otimes 2} \to g$ which satisfies:
1) $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2) Super Jacobi Identity: $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where $h$ is the Cartan subalgebra of $g$.
- Elements of $\mathfrak{H}^*$ are called weights.
- The algebra can be described by generators and relations using a Cartan matrix.
- There are two families of super Lie algebras of type I: $sl(m, n)$ and $osp(2, 2n)$.

# Super Lie algebras of type I

**Definition**

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,\ ] : g^{\otimes 2} \to g$ which satisfies:
1) $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2) Super Jacobi Identity: $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where $h$ is the Cartan subalgebra of $g$.
- Elements of $\mathfrak{H}^*$ are called weights.
- The algebra can be described by generators and relations using a Cartan matrix.
- There are two families of super Lie algebras of type I: $sl(m, n)$ and $osp(2, 2n)$.

# Super Lie algebras of type I

### Definition

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,]\ : g^{\otimes 2} \to g$ which satisfies:
1) $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2) Super Jacobi Identity: $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where $h$ is the Cartan subalgebra of $g$.
- Elements of $\mathfrak{H}^*$ are called weights.
- The algebra can be described by generators and relations using a Cartan matrix.
- There are two families of super Lie algebras of type I: $sl(m, n)$ and $osp(2, 2n)$.

# Super Lie algebras of type I

**Definition**

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,]\ : g^{\otimes 2} \to g$ which satisfies:
1) $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2) Super Jacobi Identity: $[x,[y,z]] = [[x,y],z] + (-1)^{\bar{x}\bar{y}}[y,[x,z]]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where $h$ is the Cartan subalgebra of $g$.
- Elements of $\mathfrak{H}^*$ are called weights.
- The algebra can be described by generators and relations using a Cartan matrix.
- There are two families of super Lie algebras of type I: $sl(m,n)$ and $osp(2,2n)$.

# Super Lie algebras of type I

## Definition

A super Lie algebra is a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\ ,]\ : g^{\otimes 2} \to g$ which satisfies:
1) $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$
2) Super Jacobi Identity: $[x, [y, z]] = [[x, y], z] + (-1)^{\bar{x}\bar{y}}[y, [x, z]]$

- There is a splitting $g = n_- \oplus \mathfrak{H} \oplus n_+$ where $h$ is the Cartan subalgebra of $g$.
- Elements of $\mathfrak{H}^*$ are called weights.
- The algebra can be described by generators and relations using a Cartan matrix.
- There are two families of super Lie algebras of type I: $sl(m, n)$ and $osp(2, 2n)$.

# Reprezentation theory of $g$

**Theorem**

There is the following correspondence:

$$\{irred.\ f.\ dimensional\ g-modules\} \longleftrightarrow highest\ weights \longleftrightarrow \Lambda = \mathbb{N}^{r-1} \times \mathbb{C}$$

$V(\lambda)$ $\quad\quad$ $\lambda$ $\quad\quad$ $((\lambda(h_i)), \lambda(h_s))$

$-typical$

$-atypical$ $\quad\quad$ $\hookrightarrow \mathbb{N}^{r-1} \times \mathbb{Z}$

# The quantization $U_h(g)$

**Definition**

Let $g$ be a super Lie algebra of type I. The quantization of $g$, denoted by $U_h(g)$ is the $\mathbb{C}[[h]]$-super-algebra generated by three families of elements $h_i$, $E_i$ and $F_i$, for $i \in \{1, ..., r\}$ with the relations:

$[h_i, h_j] = 0 \qquad [E_i, F_j] = \delta ij \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$

$[h_i, E_j] = a_{ij} E_j \qquad [h_i, F_j] = -a_{ij} F_j \qquad {E_s}^2 = {F_s}^2 = 0$

and quantum Serre type relations, where $[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$.

**Definition**

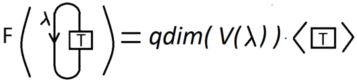
An $U_h(g)$-module $W$ is called topologically free of finite rank if there is a finite dimensional $g$-module $V$ with $W \simeq V[[h]]$ as $\mathbb{C}[[h]]$-modules.

**Theorem**

Denote by $\mathcal{M}$=the category of topologically free of finite rank $U_h(g)$-modules. Then this is a Ribbon category.

# The modified quantum dimension

- Once we obtained the Ribbon Category $\mathcal{M}$, we might think to apply the Reshetikhin-Turaev construction for that in order to obtain $\mathcal{M}-$colored link invariants.
- From the functoriality of F, we have that:



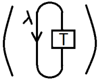

- From an argument using Kontsevich integral, it follows that: $qdim(\tilde{V(\lambda)}) = 0$ for any typical color $\lambda$.
- As a conclusion, the Reshetikhin-Turaev invariant $F(L) = 0$ for any link $L$ colored with at least one typical color.

**Idea**

Essentially, here the quantum dimension can be viewed as a function $qdim : \{weights\} \longrightarrow C[\,[\,h]\,]$ .
The main point is to replace this quantum dimension with another function such that and with a similar definition to be able to obtain link invariants.

# The modified quantum dimension

- Once we obtained the Ribbon Category $\mathcal{M}$, we might think to apply the Reshetikhin-Turaev construction for that in order to obtain $\mathcal{M}-$colored link invariants.
- From the functoriality of F, we have that:

F  $= qdim(V(\lambda)) \cdot \langle \boxed{T} \rangle$



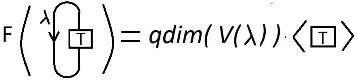
- From an argument using Kontsevich integral, it follows that: $qdim(V\tilde{(}\lambda)) = 0$ for any typical color $\lambda$.
- As a conclusion, the Reshetikhin-Turaev invariant $F(L) = 0$ for any link $L$ colored with at least one typical color.

**Idea**

Essentially, here the quantum dimension can be viewed as a function $qdim : \{weights\} \longrightarrow C[\,[\,h]\,]$ .
The main point is to replace this quantum dimension with another function such that and with a similar definition to be able to obtain link invariants.

# The modified quantum dimension

- Once we obtained the Ribbon Category $\mathcal{M}$, we might think to apply the Reshetikhin-Turaev construction for that in order to obtain $\mathcal{M}-$colored link invariants.
- From the functoriality of F, we have that:




- From an argument using Kontsevich integral, it follows that: $qdim(\tilde{V(\lambda)}) = 0$ for any typical color $\lambda$.
- As a conclusion, the Reshetikhin-Turaev invariant $F(L) = 0$ for any link $L$ colored with at least one typical color.

**Idea**

Essentially, here the quantum dimension can be viewed as a function $qdim : \{weights\} \longrightarrow C[\,[\,h]\,]$ .
The main point is to replace this quantum dimension with another function such that and with a similar definition to be able to obtain link invariants.

- In the paper "Multivariable link invariants arising from super Lie algebras of type I", N. Geer and B. Patureau defined a function $d : \{typical\ weights\} \to \mathbb{C}[\,[\,h]\,]\,[\,h^{-1}]$ called "renormalized quantum dimension" and use this as a replacement of the quantum dimension of a module in the previous setting.
- More specifically the definition would be in the following way:

**DEFINITION**

Let $L$ be a $\mathcal{M}$-colored link with at least one typical color $\lambda$. The Geer and Patureau renormalized function $F'$ is defined as:

$$F'(L) = d(\lambda) < T_\lambda >$$

where $T_\lambda$ is the tangle obtained from $T$ by cutting the $\lambda$-colored strand.

- One point that is important about that function is the fact that it should rise to link invariants. This would mean that $F'$ should not depend on the cutting strand colored with a typical color.

- In the paper "Multivariable link invariants arising from super Lie algebras of type I", N. Geer and B. Patureau defined a function $d : \{typical\ weights\} \to \mathbb{C}[[h]][h^{-1}]$ called "renormalized quantum dimension" and use this as a replacement of the quantum dimension of a module in the previous setting.
- More specifically the definition would be in the following way:

### DEFINITION

Let $L$ be a $\mathcal{M}$-colored link with at least one typical color $\lambda$. The Geer and Patureau renormalized function $F'$ is defined as:

$$F'(L) = d(\lambda) < T_\lambda >$$

where $T_\lambda$ is the tangle obtained from $T$ by cutting the $\lambda$-colored strand.

- One point that is important about that function is the fact that it should rise to link invariants. This would mean that $F'$ should not depend on the cutting strand colored with a typical color.

- In the paper "Multivariable link invariants arising from super Lie algebras of type I", N. Geer and B. Patureau defined a function $d : \{typical\ weights\} \to \mathbb{C}[\,[\,h]\,]\,[\,h^{-1}]$ called "renormalized quantum dimension" and use this as a replacement of the quantum dimension of a module in the previous setting.
- More specifically the definition would be in the following way:

**Definition**

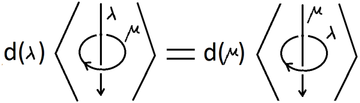
Let $L$ be a $\mathcal{M}$-colored link with at least one typical color $\lambda$. The Geer and Patureau renormalized function $F'$ is defined as:

$$F'(L) = d(\lambda) < T_\lambda >$$

where $T_\lambda$ is the tangle obtained from $T$ by cutting the $\lambda$-colored strand.

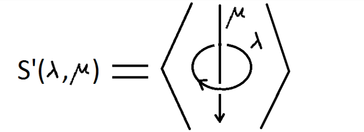
- One point that is important about that function is the fact that it should rise to link invariants. This would mean that $F'$ should not depend on the cutting strand colored with a typical color.

- Let us look at the simplest example of a link, namely the Hopf Link. Consider it colored with two typical colors $\lambda, \mu$. We would like $F'$ to be the same either if we use the cutting strand $\lambda$ or $\mu$. This is equivalent with:




- The previous relation motivates the following notation:

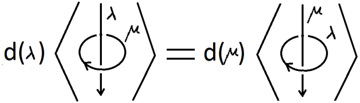
**Definition**




- This means that a necessary condition for $d$ would be:

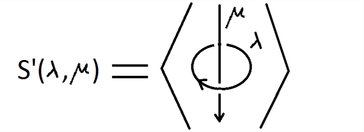
$$\frac{d(\lambda)}{d(\mu)} = \frac{S'(\lambda,\mu)}{S'(\mu,\lambda)}.$$

- Let us look at the simplest example of a link, namely the Hopf Link. Consider it colored with two typical colors $\lambda, \mu$. We would like $F'$ to be the same either if we use the cutting strand $\lambda$ or $\mu$. This is equivalent with:




- The previous relation motivates the following notation:

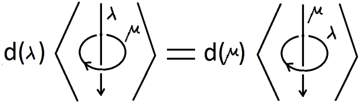
**Definition**




- This means that a necessary condition for $d$ would be:

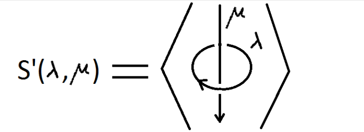
$$\frac{d(\lambda)}{d(\mu)} = \frac{S'(\lambda,\mu)}{S'(\mu,\lambda)}.$$

- Let us look at the simplest example of a link, namely the Hopf Link. Consider it colored with two typical colors $\lambda, \mu$. We would like $F'$ to be the same either if we use the cutting strand $\lambda$ or $\mu$. This is equivalent with:




- The previous relation motivates the following notation:

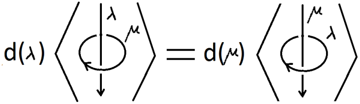
**Definition**




- This means that a necessary condition for $d$ would be:

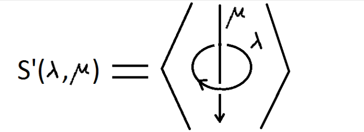
$$\frac{d(\lambda)}{d(\mu)} = \frac{S'(\lambda,\mu)}{S'(\mu,\lambda)}.$$

- Let us look at the simplest example of a link, namely the Hopf Link. Consider it colored with two typical colors $\lambda, \mu$. We would like $F'$ to be the same either if we use the cutting strand $\lambda$ or $\mu$. This is equivalent with:




- The previous relation motivates the following notation:

**Definition**




- This means that a necessary condition for $d$ would be:

$$\frac{d(\lambda)}{d(\mu)} = \frac{S'(\lambda,\mu)}{S'(\mu,\lambda)}.$$

**Proposition**

Using the character formulas for $g$-modules, there is the following relation:

$$S'(\lambda,\mu)=\frac{\varphi_{\mu+\rho}(L_1')}{\varphi_{\mu+\rho}(L_0')}\cdot f(\lambda,\mu),$$

where $f$ is a function which is symmetric in $\lambda$ and $\mu$.

- This means that the renormalized quantum dimension $d$ should verify:

$$\frac{d(\mu)}{d(\lambda)}=\frac{\frac{\varphi_{\mu+\rho}(L_0')}{\varphi_{\mu+\rho}(L_1')}}{\frac{\varphi_{\lambda+\rho}(L_0')}{\varphi_{\lambda+\rho}(L_1')}}$$

### Theorem Geer-Patureau 2010

Define $d : \{typical\ weights\} \to \mathbb{C}[[h]][h^{-1}]$ called the renormalized quantum dimension:

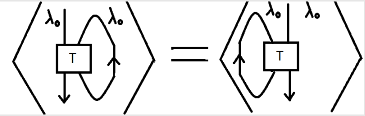
$$d(\lambda) = \frac{\varphi_{\lambda+\rho}(L'_0)}{\varphi_{\lambda+\rho}(L'_1)\varphi_{\rho}(L'_0)}.$$

Let L be a colored link with at least one typical color $\lambda$ and set $F'(L) = d(\lambda) < T_\lambda >$, where $T_\lambda$ is obtained from $T$ by cutting the $\lambda$-strand. Then $F'$ is a well defined invariant for $\mathcal{M}$-colored links colored with at least one typical color.

- We will outline a sketch of the proof:

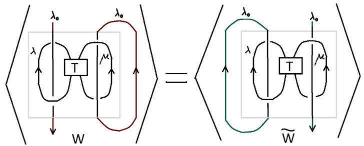
### Lemma 1

There exist a special color $\lambda_0$ such that $\forall T \in \mathcal{T}((\tilde{V}(\lambda_0), \tilde{V}(\lambda_0)))$:



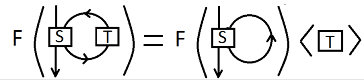
## Lemma 2

As an immediate consequence of *Lemma*1, we have:

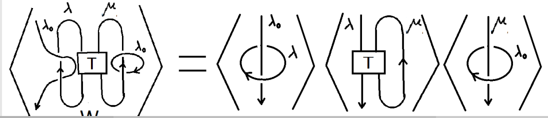



## Observation

From the monoidality of the Reshetikhin-Turaev functor, it follows that:



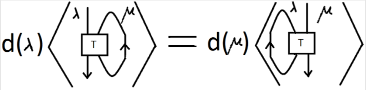

## Lemma 3



# End of the proof

## Final Lemma

For any two typical weights $\lambda$ and $\mu$ we have:




- This previous relation shows that $F'$ does not depends on the cutting strand so it concludes the well definition of the renormalized construction.

- We just defined invariants for links, but which have values almost in $\mathbb{C}[[h]]$. The next theorem shows that in fact they have in some sense one polynomial behavior once we fix the semicolors parametrized by $\mathbb{N}^{r-1}$ and we allow the last complex numbers to vary.

**Theorem (Geer and Patureau)**

Consider $L$ a link with k components which are ordered and colored with elements $\bar{c}_i \in \mathbb{N}^{r-1}$. Denote by $\bar{c} = (\bar{c}_1, ..., \bar{c}_k)$. Then there is a Laurent polynomial in many variables $M(L, \bar{c})$ such that:
1)

$$M(L,\bar{c}) \in \begin{cases} M_1^{\bar{c}_1}(q,q_1)^{-1}\mathbb{Z}[q^{\pm 1}, {q_1}^{\pm 1}] & \text{if } k=1 \\ \mathbb{Z}[q^{\pm 1}, {q_1}^{\pm 1}, ..., {q_k}^{\pm 1}] & \text{if } k \geq 2 \end{cases}$$

2)For $L'$ a framing on $L$ and $(\xi_1, ..., \xi_k) \in \mathbb{T}_{\bar{c}_1} \times ... \times \mathbb{T}_{\bar{c}_k}$, if we color the $i'th$ knot from $L'$ with $\tilde{V}_{\xi_i}^{\bar{c}_i}$ then:

$$F'(L') = e^{\sum lk_{i,j} < \lambda_{\xi_i}^{\bar{c}_i}, \lambda_{\xi_j}^{\bar{c}_j} + 2\rho > \frac{h}{2}} \; M(L, \bar{c}) \mid_{q_i = e^{\frac{\xi_i h}{2}}} .$$

# Relations with other known invariants

- The importance of these multivariable polynomial invariants can be seen from the fact that they are strongly related with other previously known invariants of polynomial type.
- First of all, one specialization of the renormalized invariants $M^{(0,\ldots,0)}_{sl(m|1)}$ recovers the multivariable Alexander polynomials.
- Moreover, the multivariable invariants recover the ADO (Akutsu, Deguchi and Ohtsuki) invariants and they are a generalization of the invariants defined by Links and Gould.
- Also, $\{M^{(0,\ldots,0)}_{sl(m|1)}\}_{m\geq 2}$ have non-trivial intersection with the HOMFLY-PT polynomials and this intersection contains the Kashaev invariants.

# Relations with other known invariants

- The importance of these multivariable polynomial invariants can be seen from the fact that they are strongly related with other previously known invariants of polynomial type.
- First of all, one specialization of the renormalized invariants $M^{(0,\dots,0)}_{sl(m|1)}$ recovers the multivariable Alexander polynomials.
- Moreover, the multivariable invariants recover the ADO (Akutsu, Deguchi and Ohtsuki) invariants and they are a generalization of the invariants defined by Links and Gould.
- Also, $\{M^{(0,\dots,0)}_{sl(m|1)}\}_{m\geq 2}$ have non-trivial intersection with the HOMFLY-PT polynomials and this intersection contains the Kashaev invariants.

# Relations with other known invariants

- The importance of these multivariable polynomial invariants can be seen from the fact that they are strongly related with other previously known invariants of polynomial type.
- First of all, one specialization of the renormalized invariants $M^{(0,\ldots,0)}_{sl(m|1)}$ recovers the multivariable Alexander polynomials.
- Moreover, the multivariable invariants recover the ADO (Akutsu, Deguchi and Ohtsuki) invariants and they are a generalization of the invariants defined by Links and Gould.
- Also, $\{M^{(0,\ldots,0)}_{sl(m|1)}\}_{m\geq 2}$ have non-trivial intersection with the HOMFLY-PT polynomials and this intersection contains the Kashaev invariants.

# Relations with other known invariants

- The importance of these multivariable polynomial invariants can be seen from the fact that they are strongly related with other previously known invariants of polynomial type.
- First of all, one specialization of the renormalized invariants $M^{(0,\dots,0)}_{sl(m|1)}$ recovers the multivariable Alexander polynomials.
- Moreover, the multivariable invariants recover the ADO (Akutsu, Deguchi and Ohtsuki) invariants and they are a generalization of the invariants defined by Links and Gould.
- Also, $\{M^{(0,\dots,0)}_{sl(m|1)}\}_{m\geq 2}$ have non-trivial intersection with the HOMFLY-PT polynomials and this intersection contains the Kashaev invariants.

# Further directions

- As we have seen, the renormalized construction has as an input an algebraic data, namely a super Lie algebra of type I and leads to multivariable link invariants.
- However, the methods that are used have purely algebraic and combinatorial flavors.
- An natural question would be to find a geometrical description for these multivariable invariants.

# Further directions

- As we have seen, the renormalized construction has as an input an algebraic data, namely a super Lie algebra of type I and leads to multivariable link invariants.
- However, the methods that are used have purely algebraic and combinatorial flavors.
- An natural question would be to find a geometrical description for these multivariable invariants.

# Further directions

- As we have seen, the renormalized construction has as an input an algebraic data, namely a super Lie algebra of type I and leads to multivariable link invariants.
- However, the methods that are used have purely algebraic and combinatorial flavors.
- An natural question would be to find a geometrical description for these multivariable invariants.

THANK YOU!