A COMBINATORIAL DESCRIPTION OF THE CENTRALIZER ALGEBRA CONNECTED TO THE LINKS-GOULD INVARIANT

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Abstract. In this paper we study the tensor power of a 4-dimensional representation of the quantum super-algebra $U_q(sl(2|1))$, focusing on the ring of its algebra endomorphisms denoted by LG_n. Its dimension was conjectured by I. Marin and E. Wagner [4]. We will prove this conjecture, describing the intertwiners spaces from a semi-simple decomposition as sets of certain paths in a lattice with integer coordinates. In the second part, we are interested in the study of a unitary matrix basis for this algebra, having in mind the algebra Birman-Murakami-Wenzl [2].

1. The quantum group $U_q(sl(2|1))$

2. The Links Gould invariant

3. Dimension of LG_n

3.1. Combinatorial description for the intertwiners of $V(0,\alpha)^{\otimes n}$.

As we have seen, $LG_n = End_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n})$.
In [4], there is stated a conjecture about the dimension of this space:

Conjecture 1. *(Marin-Wagner) $dim(LG_n) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$.*

We will prove this conjecture, by describing in a combinatorial way the intertwiners that occur in the tensor decomposition of $V(0,\alpha)^{\otimes n}$.

Theorem 1. *([3], Lemma 1.3) If $\alpha,\beta \in \mathbb{C}^*, n \in \mathbb{N}$ so that $V(0,\alpha)\otimes V(n,\beta)$ is semi-simple then:*
1) For $n \neq 0$:
$V(0,\alpha)\otimes V(n,\beta) = V(n,\alpha+\beta)\oplus V(n+1,\alpha+\beta)\oplus V(n-1,\alpha+\beta+1)\oplus V(n,\alpha+\beta+1)$.
2) For $n=0$: $V(0,\alpha)\otimes V(0,\beta) = V(0,\alpha+\beta)\oplus V(0,\alpha+\beta+1)\oplus V(1,\alpha+\beta)$.

Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ so that $V(0,\alpha)^{\otimes k}$ is semi-simple, $\forall k \in \{2,...,n\}$.
This certainly includes the cases where $\alpha \in \mathbb{C}\setminus\mathbb{Q}$.

Notation 1. *$V(0,\alpha)^{\otimes k} = \oplus_{x,y\in\mathbb{N}\times\mathbb{N}}(T_k(x,y)\otimes V(x,k\alpha+y))$ where $T_k(x,y)$ is the intertwiner space corresponding to the weight $(x,k\alpha+y)$.*

We will encode this in a graph in the plane with integer coordinates.

Definition 2. *We say that $D(n)$ is a diagram for $V(0,\alpha)^{\otimes n}$ if it is included in the lattice with integer coordinates and weights natural numbers such that for each point $(x,y)\in D(n)$, it has the associated multiplicity $t_n(x,y) = dimT_n(x,y)$. This encodes in the position (x,y) the multiplicity of the module with highest weight that moves from the fundamental weight $(0,n\cdot\alpha)$ with x from 0 and with y from $n\alpha$. In other words, we can think that the origin has coordinate $(0,n\cdot\alpha)$.*

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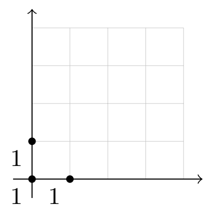
As we can see, we can deduce the tensor decomposition of $V(0,\alpha)^{\otimes n}$ by just reading the non-zero multiplicities associated to points in $D(n)$.
For example, for $n=2$:

$$V(0,\alpha)\otimes V(0,\alpha)=V(0,2\alpha)\oplus V(0,2\alpha+1)\oplus V(1,2\alpha).$$

(1)

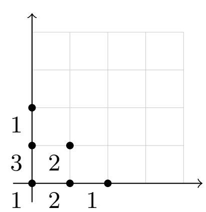


$n=3$
$V(0,\alpha)^{\otimes 3}=(V(0,2\alpha)\oplus V(0,2\alpha+1)\oplus V(1,2\alpha))\otimes V(0,\alpha)=$
$(V(0,2\alpha)\otimes V(0,\alpha))\oplus(V(0,2\alpha+1)\otimes V(0,\alpha))\oplus(V(1,2\alpha)\otimes V(0,\alpha)=$
$(V(0,3\alpha)\oplus V(0,3\alpha+1)\oplus V(1,3\alpha))\oplus(V(0,3\alpha+1)\oplus V(0,3\alpha+2)\oplus V(1,3\alpha+1))\oplus$
$(V(1,3\alpha)\oplus V(1,3\alpha+1)\oplus V(0,3\alpha+1)\oplus V(2,3\alpha))\Rightarrow$

$$V(0,\alpha)^{\otimes 3}=V(0,3\alpha)\oplus 3\cdot V(0,3\alpha+1)\oplus V(0,3\alpha+2)\oplus 2\cdot V(1,3\alpha)\oplus 2\cdot V(1,3\alpha+1)\oplus V(2,3\alpha)$$

So, $D(3)$ is:

(2)



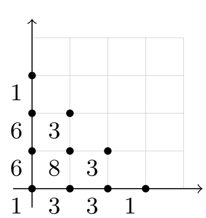
$n=4$
$V(0,\alpha)^{\otimes 4}=(V(0,\alpha)^{\otimes 3}\otimes V(0,\alpha))=((V(0,3\alpha)\otimes V(0,\alpha))\oplus 3\cdot(V(0,3\alpha+1)\otimes V(0,\alpha))\oplus$
$(V(0,3\alpha+2)\otimes V(0,\alpha))\oplus 2\cdot(V(1,3\alpha)\otimes V(0,\alpha))\oplus 2\cdot(V(1,3\alpha+1)\otimes V(0,\alpha))\oplus(V(2,3\alpha)\otimes V(0,\alpha))=$
$=(V(0,4\alpha)\oplus V(0,4\alpha+1)\oplus V(1,4\alpha))\oplus(3V(0,4\alpha+1)\oplus 3V(0,4\alpha+2)\oplus 3V(1,4\alpha+1))\oplus$
$(V(0,4\alpha+2)\oplus V(0,4\alpha+3)\oplus V(1,4\alpha+2))\oplus(2V(1,4\alpha)\oplus 2V(1,4\alpha+1)\oplus 2V(0,4\alpha+1)\oplus 2V(2,4\alpha))$
$(2V(1,4\alpha+1)\oplus 2V(1,4\alpha+2)\oplus 2V(0,4\alpha+2)\oplus 2V(2,4\alpha+1))\oplus(V(2,4\alpha)\oplus V(2,4\alpha+1)\oplus V(1,4\alpha+$
$1)\oplus V(3,4\alpha))\Rightarrow$

$$V(0,\alpha)^{\otimes 4}=V(0,4\alpha)\oplus 6\cdot V(0,4\alpha+1)\oplus 6\cdot V(0,4\alpha+2)\oplus V(0,4\alpha+3)\oplus 3\cdot V(1,4\alpha)\oplus 8\cdot V(1,4\alpha+1)\oplus 3\cdot V(1,4\alpha+2)\oplus$$

$$3\cdot V(2,4\alpha)\oplus 3\cdot V(2,4\alpha+1)\oplus V(3,4\alpha)$$

We obtain $D(4)$:

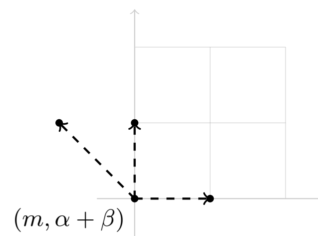
(3)



In the sequel, we will describe that the diagrams $D(n)$, can be constructed recursively, more precisely, if we know $D(n)$, then by applying some moves we will be able to obtain $D(n+1)$.
Let us start with $V(m,\beta)$. We will encode the decomposition of $V(m,\beta)\otimes V(0,\alpha)$ in a lattice. Let us think that initially, $V(m,\beta)$ is encoded by diagram D which has as origin (m,β) and the corresponding multiplicity 1.

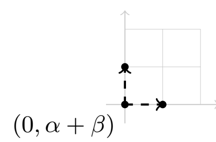
Definition 3. *a) From the Theorem 1, we have:*
$V(0,\alpha)\otimes V(m,\beta) = V(m,\alpha+\beta)\oplus V(m+1,\alpha+\beta)\oplus V(m-1,\alpha+\beta+1)\oplus V(m,\alpha+\beta+1)$.
We call the effect of tensoring $V(m,\beta)$ with $V(0,\alpha)$ a blow up of type (m,β) and $B(m,\beta)$ the new corresponding diagram.

(4)



b) $V(0,\alpha)\otimes V(0,\beta) = V(0,\alpha+\beta)\oplus V(0,\alpha+\beta+1)\oplus V(1,\alpha+\beta)$.
We call the effect of tensoring $V(0,\beta)$ with $V(0,\alpha)$ a blow up of type $(0,\beta)$ and $B(0,\beta)$ the new corresponding diagram.

(5)



Lemma 4. *The diagram $D(n+1)$ can be obtained from $D(n)$, by blowing up each point $(x,y)\in D(n)$ with $B(x,y)$ for $t_n(x,y)$ times and add in each vertex all the new multiplicities.*

Proof. Suppose we have $D(n)$. This means that:

$$V(0,\alpha)^{\otimes n} = \oplus_{x,y\in\mathbb{N}\times\mathbb{N}}(t_n(x,y)\cdot V(x,n\alpha+y))$$

In order to deduce the multiplicities that occur in $D(n+1)$, we have:

$$V(0,\alpha)^{\otimes n+1} = \oplus_{x,y\in\mathbb{N}\times\mathbb{N}}(t_n(x,y)\cdot(V(x,n\alpha+y)\otimes V(0,\alpha)))\quad(*)$$

On the other hand, we are interested in the multiplicities t_{n+1}, where:

$$V(0,\alpha)^{\otimes n+1} = \oplus_{x,y\in\mathbb{N}\times\mathbb{N}}(t_{n+1}(x,y)\cdot V(x,n+1\alpha+y))$$

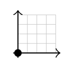
In the previous description $(*)$, $(V(x,n\alpha+y)\otimes V(0,\alpha))$ will add a blow up of center
$(x+0,(n\alpha+y)+\alpha) = (x,(n+1)\alpha+y)$, which is encoded in $D(n+1)$, as a blow-up $B(x,y)$ with center (x,y). For each point (x,y), we'll have to do the bow-up $t_n(x,y)$ times.
In this way, we obtain $t_{n+1}(x,y)$. □

Up to this point, we saw how to construct the recursive relation with step one $D(n)\to D(n+1)$. However, this is still at a theoretical point of view. We know the diagram $D(2)$, and in the following part, we will show how each $t_n(x,y)$ can be describes in a natural way using paths in the plane.

Remark 5. *1)In $D(n+1)$, for each point (x,y), the total multiplicity if obtained by adding all the multiplicities of the points from $D(n)$, which can arrive to (x,y) using one of the following moves:*

M1)	*stay move*	$(x,y) \to (x,y)$
M2)	$\longrightarrow$	$(x,y) \to (x+1,y)$
M3)	$\uparrow$	$(x,y) \to (x,y+1)$
M4)	$\nwarrow$	$(x,y) \to (x-1,y+1)$ *if* $x>0$.

Here, the reason for the fact that M_4 can be done just if $x>0$ is that that occurs in the blow-up $B(x,y)$ if and only if $x>0$.
2)If we start from $D(n-1)$, we can obtain $D(n+1)$, by counting all the paths in the integer lattice (with the corresponding multiplicities as in $D(n-1)$.
In this way: $t_{n+1}(x,y)$ is the sum of all paths of length 2 starting from points in $D(n-1)$ and ending in (x,y) with the moves M_1,M_2,M_3 or M_4.
3)Iterating this argument by Induction and using the fact that $D(1)$ is:

(6) 

1

we obtain the following combinatorial description for the intertwiners spaces:

Theorem 6. *In $D(n)$, for each point $(x,y) \in \mathbb{Z}\times\mathbb{Z}$, we have:*
$t_n(x,y)$ =number of paths from (0,0) to (x,y) with (n-1) steps and moves M1, M2, M3 or M4 with the condition that they do not have any point with a negative coordinate on the x-axis.

Remark 7. *In $D(n)$, just the points that are in the standard simplex of lenght $n-1$: Δ_{n-1} have non-zero weights.*

Notation 2. *$P_n(x,y) :=$ paths from $(0,0)$ to (x,y) with $(n-1)$ steps and moves M1, M2, M3 or M4 with the condition that they do not have any point with a negative coordinate on the x-axis*

Remark 8.

$$V(0,\alpha)^{\otimes n} = \oplus_{x,y\in\mathbb{N}\times\mathbb{N}}(t_n(x,y)V(x,n\alpha+y))$$

where $t_n(x,y)$ is the cardinality of the intertwiner space corresponding to the weight $(x,k\cdot\alpha+y)$.
From [3], *for typical (n,α), $V(n,\alpha)$ is simple and $Morph_{U_q(sl(2|1))}(V(n,\alpha),(m,\beta)) \simeq \delta_{(n,\alpha)}^{(m,\beta)}\mathbb{C}\cdot Id$.*
In our case, all $V(x,n\alpha+y)$ are typical and we have:
$End_{U_q(sl(2|1))}(V(0,\alpha)^{\otimes n}) \simeq \oplus End_{U_q(sl(2|1))}(t_n(x,y)V(x,n\alpha+y))) \simeq \oplus M(t_n(x,y),\mathbb{C})$.

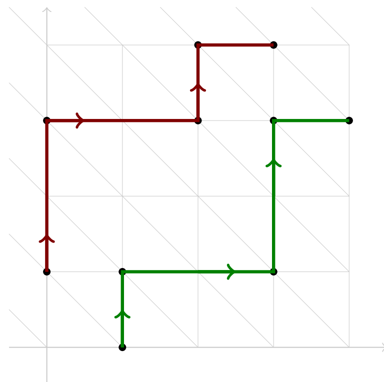
Corollary 9. $dimLG_n = \sum_{x,y\in\mathbb{N}\times\mathbb{N},x+y\leq n-1} t_n(x,y)^2$.

3.2. **Computation for** $dim\ LG_n$**.**

In [4], it is mentioned that F.Chapoton remarked that conjectured dimension of LG_{n+1} coincides with a combinatorial quantity:

Theorem 10. [1], [5], $\frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$ *=number of pairs of paths in the $(n+1)\times(n+1)$ square which go $\uparrow$ or $\rightarrow$, between $(0,1)\to(n,n+1)$ and $(1,0)\to(n+1,n)$ which do not intersect.*

We will denote by C_{n+1} this set of pairs of paths. We will prove the Conjecture using the previous description for $dim\ LG_{n+1}$ and this result for the conjectured number.
Since in $dim\ LG_{n+1}$, there are counted all $t_{n+1}(x,y)$, $(x,y)\in{\Delta_n}^{\mathbb{Z}}$, we will describe C_{n+1} as a sum indexed by the same set. Having a pair of paths in the square, we can remember where those "cut the diagonal", and use that as an indexing set.

(7) 

Definition 11. *For* $(a, b) \in \mathbb{N} \times \mathbb{N}$ *with* $a, b \leq n+1$ *and* $a > b$*, denote by*
1)$C_{n+1}(a, b) :=$*pairs of paths in* C_{n+1} *that cut the diagonal of the square precisely in* $(n+1-a, a), (n+1-b, b)$.
2)$C^{\Delta}{}_{n+1}(a, b) :=$*pairs of disjoint paths in* Δ_{n+1} *between* $(1, 0) \to (n+1-a, a), (0, 1) \to (n+1-b, b)$.

Remark 12.

$$1)\ C_{n+1} = \cup_{a,b \leq n+1, a>b} C_{n+1}(a, b)$$

$$2)\ C_{n+1}(a, b) \simeq C^{\Delta}{}_{n+1}(a, b) \times C^{\Delta}{}_{n+1}(a, b)$$

by just cutting a path from $C_{n+1}(a, b)$ *at the diagonal, and obtaining two paths in* $C^{\Delta}{}_{n+1}(a, b)$.

Proposition 13. *From the previous remarks and definitions we obtain that:*

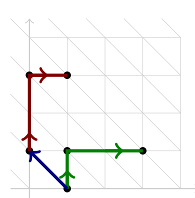
$$C_{n+1} = \cup_{(a,b \leq n+1), a>b} (C^{\Delta}{}_{n+1}(a, b) \times C^{\Delta}{}_{n+1}(a, b))$$

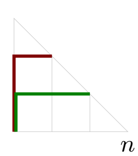
$$\mid C_{n+1} \mid = \sum_{a,b \leq n+1, a>b} \mid C^{\Delta}{}_{n+1}(a, b) \mid^2$$

Notation 3. *For* $(a, b) \in \mathbb{N} \times \mathbb{N}$ *with* $a, b \leq n$ *and* $a \geq b$*, denote by*
$D^{\Delta}{}_{n}(a, b) :=$*pairs of paths in* Δ_n *between* $(0, 0) \to (n+1-a, a), (0, 0) \to (n+1-b, b)$ *that can intersect each other just in integer points, but they do not cross each other.*

Proposition 14. $C^{\Delta}{}_{n+1}(a, b) \simeq D^{\Delta}{}_{n}(a, b-1)$

Proof. Let $C, D \in C^{\Delta}{}_{n+1}(a, b)$ pair of paths.
By modifying $C \to C + (-1, 1)$, we will obtain $C + (-1, 1), D \in D^{\Delta}{}_{n}(a, b-1)$
(where here the simplex Δ_n is seen as bounded by $(0, 1), (n, 1), (n+1, 0)$). After that it can be easily shown that this function is a bijection. $\square$

(8) 

(9) 

Proposition 15. *From the last bijection, we can deduce that we can count C_n using pairs of paths in Δ_n:*

$$| C_{n+1} |= \sum_{a,b\leq n; a\geq b} | D^{\Delta}{}_n(a,b) |^2$$

Remark 16. *For any $C_1 = ((C_1)^k_x, (C_1)^k_y), C_2 = ((C_2)^k_x, (C_2)^k_y) \in D^{\Delta}{}_n(a,b)$ the condition that they do not cross each other can be read as:*
for any step k:$(C_1)^k_y \leq (C_2)^k_y$.

Now, we arrive at the last part, and we will show a correspondence between $t_n(x,y)$ and $D^{\Delta}{}_n(a,b)$.

Lemma 17. *We have the following correspondence for any $(x_0, y_0) \in \Delta_n$:*
$P_n(x_0, y_0) \simeq D^{\Delta}{}_n(x_0 + y_0, y_0)$

Proof. Let $C_1, C_2 \in D^{\Delta}{}_n(x+y, y)$. This pair of paths can be encoded in a sequence of moves of four types.
For $(x_1, y_1) \in C_1, (x_2, y_2) \in C_2$ the k'th step, in order to pass to the $k+1$ step we have four situations.
Movements: $((x_1, y_1),\ (x_2, y_2))$ and we know $y_2 \geq y_1$
$((0,1),\ (0,1))$
$((1,0),\ (1,0))$
$((1,0),\ (0,1))$
$((0,1),\ (1,0))$
On the other hand, for any path $C \in t_n(x,y)$, this can be encoded also, by specifying which move we do from the k^th step to the $(k+1)^st$:
Movements: (x,y) and we know $x \geq 0$
$(0,0)$
$(0,1)$
$(1,0)$
$(-1,1)$
Now, we want to define a function $f : D^{\Delta}{}_n(x_0 + y_0, y_0) \to t_n(x_0, y_0)$.
Let $C_1, C_2 \in D^{\Delta}{}_n(x+y, y)$. We want to sent each pair of points $(x_1, y_1) \in C_1, (x_2, y_2) \in C_2$ in $f((x_1, y_1), (x_2, y_2)$ such that it
satisfies the restrictions from $t_n(x_0, y_0)$. Since we know the condition $y_2 \geq y_1$, it would be natural to send $f((x_1, y_1), (x_2, y_2))_1 = y_2 - y_1$
which would ensure us the necessary condition.
Consider $f((x_1, y_1), (x_2, y_2)) := (y_2 - y_1, x_2)$.
Then $f((0,0), (0,0)) = (0,0)$, so it preserves the initial points. Now we can verify that this transformation, preserves correspondingly the possible moves in the two cases in the following way:

$(x_1, y_1),\ (x_2, y_2) \longrightarrow$	$(y_2 - y_1, x_2)$
$(0,1),\ (0,1)$	$(0,0)$
$(1,0),\ (1,0)$	$(0,1)$
$(1,0),\ (0,1)$	$(1,0)$
$(0,1),\ (1,0)$	$(-1,1)$

This concludes that f is a well-defined bijection.

□

As a conclusion, from Corollary 9, Theorem 10, Lemma 17 we obtain the Wagner-Marin Conjecture:

Theorem 18.

$$dim(LG_n) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}.$$

4. Describing an unitary matrix basis for LG_n

In [2], C. Blanchet and A. Beliakova described a basis for the algebra Birman-Murakami-Wenzl by idempotents elements. We will try to study elements from a basis of LG_n using similar methods.

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