A Homological model for the coloured Jones polynomials

Cristina Ana-Maria Palmer-Anghel

Advisor: Prof. Christian Blanchet — Université Paris Diderot

Quantum invariants for knots

The theory of quantum invariants began in 1987 with the Jones polynomial and continued with a general method due to Reshetikhin and Turaev that starting with any Ribbon category leads to link invariants. This method is purely algebraic and combinatorial. The coloured Jones polynomials are obtained from this construction using representation theory of $U_q(sl(2))$.

The coloured Jones polynomials

Proposition: $Rep(U_q(sl(2)))$ is a ribbon category, such that $\forall V, W \in Rep(U_q(sl(2)))$ it has the following compatible morphisms:
1) A braiding that comes from the R-matrix: $R_{V,W} : V \otimes W \to W \otimes V$
2) Dualities: $coev_V : \mathbb{Z}[q^{\pm}] \to V \otimes V^* \quad ev_V : V \otimes V^* \to \mathbb{Z}[q^{\pm}]$

Definition: The R-matrix of $U_q(sl(2))$ leads to a sequence of a braid group representations: $\varphi_n^N : B_n \to End_{U_q(sl(2))}(V_N^{\otimes n})$

$$\sigma_i^{\pm} \longleftrightarrow Id_V^{(i-1)} \otimes R^{\pm}_{V_N,V_N} \otimes Id_V^{(n-i-1)}$$

Definition: Coloured Jones polynomials
The N^{th} coloured Jones polynomial $J_N(L,q) \in \mathbb{Z}[q^{\pm}]$ is constructed in from a diagram of the link L, by composing the morphisms at each crossing, cup and cap in the following way:

$$F(\times) = R_{V_N,V_N} \quad F(\cup) = coev_{V_N} \quad F(\cap) = ev_{V_N}$$

Motivation-Homological interpretations:

Jones polynomial: For $N=2$, $J_2(L,q)$ is the original Jones polynomial. This is a quantum invariant but can be defined also by skein relations. In 1994, Bigelow[Big02] and Lawrence[Law93] described geometrically the Jones polynomial as a graded intersection pairing between homology classes in a covering of a configuration space using its skein nature for the proof.
Motivation-Homological interpretations: As we have seen, the definition of the coloured Jones polynomials is purely algebraic.
Aim: We will describe a topological interpretation for $J_N(L,q)$.
Unlike the original Jones polynomial, the coloured Jones polynomials do not have a direct definition by skein relations. For our model, we use the definition of $J_N(L,q)$ as a quantum invariant and study more deeply the Reshetikhin-Turaev functor that leads to this invariant.

Quantum representation

For $n, m \in \mathbb{N}$, there is a subspace $W^N_{n,m} \subseteq V_N^{\otimes n}$ called the highest weight space of the module V_N. The braid group action φ_n^N preserves $W^N_{n,m}$, defining the quantum representation : $\varphi^N_{n,m} : B_n \to Aut(W^N_{n,m}; \mathbb{Z}[q^{\pm 1}])$

Strategy

1) Let L be a link and consider a braid $\beta \in B_{2n}$ such that $L = \hat{\hat{\beta}}$ (plat closure). We will study the Reshetikhin-Turaev construction more deeply at 3 main levels:
1) union of cups ∩∩∩∩
2) braid β
3) union of caps ∪∪∪∪
We start with $1 \in \mathbb{Z}[q^{\pm 1}]$ and $J_N(L,q) = F(L)(1)$.
The important remark is the fact that even if, a priori, after the level 2) we have the associated morphism $\varphi_n^N(\beta) \in End(V_N^{\otimes n})$, actually coming from level 3) we arrive in the highest weight space $W_{2n,n(N-1)}$.
2) From the invariance of $W_{2n,n(N-1)}$ with respect to the braid group action, we conclude that we can obtain the coloured Jones polynomial by doing the whole construction through these subspaces.
3) The importance of this step is related to the fact that Kohno proved in 2012 that there exists a homological counterpart for the highest weight spaces, which are the Lawrence representations.

References

- [Big02] Stephen Bigelow, *A homological definition of the Jones polynomial*, Geometry & Topology Monographs 4 (2002) 29–41.
- [Koh12] Toshitake Kohno- Quantum and homological representations of braid groups. Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale (2012), 355–372.

Lawrence representation

In 1990, R. Lawrence introduced a sequence of homological representations of the braid group B_n. Let $C_{n,m}$ be the unordered configuration space on the n-punctured disc $\mathbb{D}_n$. Consider a local system $\varphi : \pi_1(C_{n,m}) \to \mathbb{Z}\langle x \rangle \oplus \mathbb{Z}\langle d \rangle$ and $\tilde{C}_{n,m}$ the corresponding cover. The homology of $\tilde{C}_{n,m}$ will be a $\mathbb{Z}[x^{\pm}, d^{\pm}]$-module.
Let $H_{n,m} \subseteq H^{lf}_m(\tilde{C}_{n,m}, \mathbb{Z})$ be the subspace generated by certain classes of submanifolds called multiforks. Since $B_n = MCG(\mathbb{D}_n)$, it induces an action on the homology of the covering which preserves $H_{n,m}$, called the Lawrence representation: $l_{n,m} : B_n \to \mathrm{Aut}(H_{n,m}, \mathbb{Z}[x^{\pm}, d^{\pm}])$.

Identification between quantum and homological representations

Specialisation: Consider the following specialisation of the coefficients:

$$\psi_N : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Z}[q^{\pm}] \quad \psi_N(x) = q^{2(N-1)}; \psi_N(d) = -q^{-2}.$$
$$H_{n,m}|_{\psi_N} := H_{n,m} \otimes_{\psi_N} \mathbb{Z}[q^{\pm}]$$

Bigger highest weight spaces: Let $\hat{W}^N_{n,m} \subseteq \hat{V}^{\otimes n}_{N-1}$ be the highest weight space of the Verma module for $U_q(sl(2))$, which has the B_n-action $\hat{\varphi}^N_{n,m}$.
We remark that we have: $W^N_{n,m} \subset \hat{W}^N_{n,m}$.
Theorem(Kohno)[Koh12]: The quantum and homological representations of the braid group are isomorphic:

$$\hat{\varphi}^N_{n,m} \curvearrowright \hat{W}^N_{n,m} \simeq H_{n,m}|_{\psi_N} \curvearrowleft l_{n,m}|_{\psi_N}$$

Problem: This model gives a homological interpretation for the big highest weight spaces. In our model we have the smaller $W_{2n,n(N-1)}$ subspaces involved.

Strategy

4) We prove that we can do the construction of $J_N(L,q)$ through the bigger highest weight spaces, and for that we extend the evaluation on $W^N_{n,m}$ to a kind of evaluation for $\hat{W}^N_{n,m}$.
5) Use Kohno's identification to give a homological counterpart for the braid part β (2).
6) For the coevaluation (3), we will consider a certain element in $H_{2n,n(N-1)}$ which corresponds to $coev^{\otimes n}_{V_N}$.
7) In order to give a topological counterpart for the evaluation (1), we will use a graded intersection pairing between the Lawrence representation and a "dual" space.

Blanchfield pairing

Let $H^{\partial}_{n,m} \subseteq H^{\partial}_m(\tilde{C}_{n,m}, \mathbb{Z})$ be a certain subspace generated by classes of submanifolds called barcodes. There exists a graded intersection pairing which is sesquilinear: $\langle \, , \rangle : H_{n,m} \otimes H^{\partial}_{n,m} \to \mathbb{Z}[x^{\pm}, d^{\pm}]$
Let the specialisation $\alpha_N : \mathbb{Z}[x^{\pm}, d^{\pm}] \to \mathbb{Q}(q)$ be induced from ψ_N.
Lemma (-): The specialised Blanchfield pairing is non-degenerate:

$$\langle \, , \rangle|_{\alpha_N} : H_{n,m}|_{\alpha_N} \otimes H^{\partial}_{n,m}|_{\alpha_N} \to \mathbb{Q}(q)$$

Homological model for the coloured Jones polynomials $J_N(L,q)$

Theorem (-): For $\forall \, n \in \mathbb{N}$, there exist $F \in H_{2n,n(N-1)}$ and $G \in H^{\partial}_{2n,n(N-1)}$ such that for any L link and $\beta \in B_{2n}$ such that $L = \hat{\hat{\beta}}$, the coloured Jones polynomial has the interpretation:

$$J_N(L,q) = \langle \beta F, G \rangle|_{\alpha_N}$$

Further directions

In this homological model, the homology classes F and G are given by linear combinations of Lagrangian submanifolds in the covering of the configuration space. The further question would be to study the graded Floer homology groups that come from this model and whether they lead to a well defined categorification for the coloured Jones polynomials.

References

- [Law93] R. J. Lawrence - A functorial approach to the one-variable Jones polynomial. J. Differential Geom., 37(3):689-710, 1993.

Web: www.cristinaanghel.ro Email: cristina.anghel@imj-prg.fr