Topological degree and invariance of domain theorems

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The invariance of domain theorem, appeared at the beginning of 20^{th} century. There are two approaches for the proof: the first one, uses separation theorems of Jordan-Brouwer type based on homological techniques. The second, which over the time had led to various generalizations, is based on topological degree and Borsuk type theorems.

The aim of my talk is to describe both methods of proof, and to present three modern applications: the first two are extensions of the invariance theorem using topological degree theory, the third being an application of the invariance of domain theorem itself to a biological migration-selection model.

The Jordan-Brouwer theorem and the main steps for proof
A nice application concerning the Moebius strip
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- Brouwer degree
- Leray-Schauder degree

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- An Invariance of domain theorem for countably condensing vector fields
- An application to a biological migration-selection model

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Acknowledgements

The Jordan-Brouwer theorem and the main steps for proof

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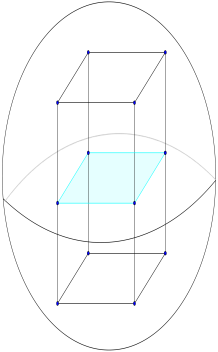
  Lemma $\tilde{H}_q(S^n \setminus e^r) = 0 \; \forall q$.
- Let z a q - cycle in $S^n \setminus e^r$. We want that z is a boundary in $S^n \setminus e^r$. The proof of the lemma is done in the following 4 steps:

First step

The complement of a fixed floor t in S^n

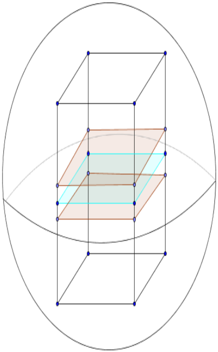


Using induction on r, we obtain that z is a boundary in $S^n \setminus e^{r-1}(t)$ for any t, where $e^{r-1}(t)$ is the image of $I^{r-1} \times \{t\}$.

Second step

Using compacteness arguments and uniform continuity of continuous functions on compact sets, we obtain that for any t there is an ϵ_t such that z is a boundary in the complement of the image of $I^{r-1} \times (t - \epsilon_t, t + \epsilon_t)$.

The complement of a neighborhood of the t floor in S^n



Third and fourth steps

- **Third step** Using Lebesgue lemma for metric spaces and the previous step, we ca split $[0,1]$ in M intervals of the form $[j/M,(j+1)/M]$ such that z is a boundary in the complement of the image of $I^{r-1}\times[j/M,(j+1)/M]$ for any j.

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- **Fourth step** Using the Mayer-Vietoris sequence and the inductive assumption on $r - 1$ we find finally that z is a boundary in $S^n \setminus e^r$.

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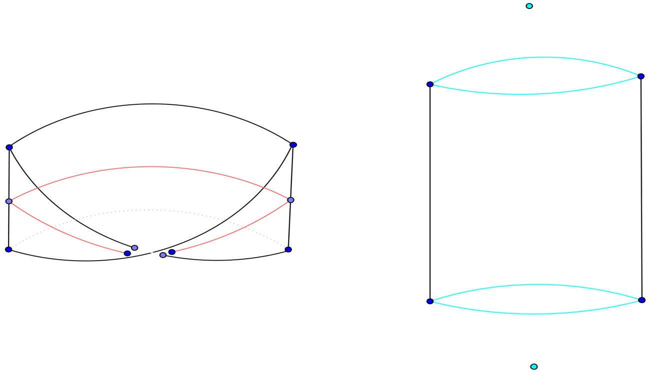
- Of course, the answer is NO! Because one is orientable and the other is not.
- Are these manifolds homeomorphic ?
- The answer is still NO, but the argument is more subtle.

The Moebius strip with median circle and the cylinder with the 2 points used for compactification



In fact if such a homeo would exists, it should send the equator E of the Moebius strip in a simple closed curve C on the cylinder. But compactifying the cylinder and applying Jordan-Brouwer theorem, C will disconnect the cylinder. This is a contradiction with the fact that E does NOT disconnect the Moebius strip.

The Invariance of the domain theorem I: Jordan-Brouwer type approach

A strong consequence of Jordan-Brouwer theorem is the following:

Theorem of the invariance of the domain Let U connected open in $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ continuous and injective. Then $f(U)$ is open and f is homeo on its image.

The main point is to compactify $\mathbf{R}^n$ to $\mathbf{S}^n$, to take a small ball B in U, and to apply the Jordan-Brouwer separation theorem for the image $f(\partial B)$.

Topological degree

The second approach to the invariance of domain theorem is based on the topological degree. This theory is an extension of the degree theory from the differentiable context. For $D \subset \mathbb{R}^n$ open with compact closure, and $f : D \to \mathbb{R}^n$ a $\mathcal{C}^1$ map, for a regular value $p \in \mathbb{R}^n$ we have the classical definition of the degree (cf. Milnor's book - Topology from...):

$$deg(f, D, p) = \Sigma sign(Jac_x f),$$

where the sum goes over all $x \in f^{-1}(p)$.

Brouwer degree

With the same notations as above, for f only a continous mapping, we define the degree of f as the one of a sufficiently close smooth function g.
That we can find such a g is granted by approximation theorems (cf. for example Hirsch book).
The theory of topological degree is based on the following theorem (cf. Cho and Chen book - Topological degree theory)

Existence of Brouwer degree

Theorem For an open and bounded $D \subset \mathbb{R}^n$, for $f : \bar{D} \to \mathbb{R}^n$ continous, and for $p \notin f(\partial D)$, there exist an integer denoted by $d(f, D, p)$ such that:

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- the degree is independent under homotopies of f.
- the degree is additive in the D variable, for disjoint D's.
- the degree is constant on any connected component of $\mathbb{R}^n \backslash f(\partial D)$.

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- **Borsuk odd degree theorem** If D open, bounded and symmetric with $0 \in D$, f is odd and $0 \notin f(\partial D)$ then $deg(f, D, 0)$ is odd. (in particular it is non-zero)

Leray-Schauder degree

In Banach spaces of infinite dimension it can be shown the a degree theory for arbitrary mappings cannot be defined. However, for compact continous operators (compact are those sending bounded subsets to relatively compact ones) there is a theory with the same properties as in the finite dimensional case, based on the following two results:

- **Brouwer's degree behavior w.r.t. dimensional cutting** Let $D \subset \mathbb{R}^n$ open and bounded. Let $g \in \mathcal{C}(\bar{D})$ and $m < n$. If $x \notin g(\partial D)$ then $deg(g, D, x) = deg(g_{|D_m}), D_m, x)$, where $D_m = D \cap \mathbb{R}^m$

- **Approximation for compact continous operators**
  For D open , bounded in a real Banach space X, $T : \bar{D} \to X$ a compact continous operator and for any $\epsilon > 0$ there is an $Y \subset X$ of finite dimension and $T' : \bar{D} \to Y$ such that $\| T'x - Tx \| \leq \epsilon$ for all $x \in \bar{D}$.

The above results allows a construction of a topological degree in infinite dimensional setting for compact continous operators: it's the Leray-Schauder degree. The main idea is to use aproximation by finite dimensional range operators and the behavior (in fact the invariance) of Brouwer degree under dimensional cutting.

The Invariance of the domain theorem II: Borsuk type approach

The Brouwer's and Leray-Schauder's topological degree theory give an alternative proof for the invariance of domain theorem and also an extension to the infinite dimensional case:

- **Invariance of domain in Banach spaces for continous and compact operators**
  Let D open in a Banach space X, and $f : D \to X$ compact, continous and locally injective. Then f is an open mapping.

- The main idea, a natural one in fact, is that the property of being an open mapping reduce to the condition that the equation in x, $f(x) = y$ has solution for y in certain ball around a point in the image. The existence of such solutions is highly tractable by the fundamental properties of the topological degree. Actually the homotopy invariance and the Borsuk odd degree theorem above play a central role in the proof.

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- However, at last in the finite dimensional case, it should be mentioned that the full details for the construction of the topological degree are in fact of the same order of complexity as the homological machine used in the proof of the Jordan-Brouwer separation theorem. The real advantage of this second approach is that it has far reaching generalizations.

Three modern applications
1. Invariance of Domain Theorem for nonlinear Fredholm maps of index zero between Banach spaces

After P. Benevieri, M. Furi, M. P. Pera,
The invariance of domain for C^1 Fredholm maps of index zero., Recent trends in nonlinear analysis, 35 - 39, Progr. Nonlinear Differential Equations Appl., 40, Birkhauser, Basel, 2000.

The generalisation presented in the above paper concern Fredholm maps of index 0 between Banach manifolds.

A bounded linear operator between Banach spaces is named Fredholm if both the kernel and the cokernel are finite dimensional.

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- A Fredholm map of index 0 between Banach manifolds is one whose Frechet differential in every point is Fredholm and has index 0.
- **Theorem** Let M and N Banach manifolds and $f : M \to N$ an injective Fredholm map of index 0. Then $f(M)$ is open in N.
- The main ingredient for the proof is the Brouwer degree mod-2 for maps between finite dimensional manifolds and its homotopy invariance. A very good reference for this is Milnor's book "Topology from differentiable viewpoint".

2. An Invariance of domain theorem for countably condensing vector fields

After I. S. Kim, The invariance of domain theorem for condensing vector fields, Topological Methods in Nonlinear Analysis 25 (2005), no. 2, 363 - 373.

Let E a Banach space and $\mathcal{M}$ a collection of subsets of E containing all precompact (i.e. with compact closure) subsets of E and closed under:
closure of convex cover
finite union
finite sum
multiplication by scalars.

- A function $\gamma : \mathcal{M} \to [0, \infty]$ is called a measure of noncompactness on E if:
  $\gamma(\bar{co}A) = \gamma(A)$
  $\gamma(A) = 0$ iff A is precompact
  $\gamma(A \cup B) = max(\gamma(A), \gamma(B))$
  $\gamma(A + B) \leq \gamma(A) + \gamma(B)$
  $\gamma(tA) =\mid t \mid \gamma(A)$.

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  $\gamma(tA) =\mid t \mid \gamma(A)$.
- In the above setting, for X a subset of E and k a nonnegative real number, a continuous map $F : X \to E$ is called countably k-condensing in the strong sense if $F(X) \in \mathcal{M}$ and $\gamma(F(\bar{co}(C))) \leq k\gamma(C)$ for each countable subset $C \subset X$ with $\bar{co}(C) \subset X$ and $C \in \mathcal{M}$.

- **Theorem** Let $X \subset E$ open, $k \in [0, 1)$ and $F : X \to E$ a countably k-condensing map in the strong sense, such that $Id - F$ is locally injective. Then $Id - F$ is an open map.(Id is the identity of X)

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- The main tools in the proof is the Leray-Schauder degree and its homotopy invariance.

3. An application to a biological migration-selection model

After R. J. Sacker, An invariance theorem for mappings, J. Difference Equ. Appl. 18 (2012), no. 1, 163-166.

Migration-selection models are aimed to describe the change of genetic material across evolution of populations. For example for two types of genes, G_1 and G_2, their concentrations x_1 and x_2 are supposed to be modified in time by a low of the following form:

$$x(t+1) = f(x(t)) \tag{1}$$

where t is a particular moment of time, $x = (x_1, x_2)$ is the vector of concentrations and f is a function which depends on the model.

Let X the set of possible concentration vectors. The fundamental question for such a model is the existence of a compact invariant set $K \subset X$ i.e. $f(K) \subset K$.

If such a set exists, the theory of systems of difference equations can be applied to produce a globally attracting fixed point $\hat{x}$. This means a point such that if we start from any concentration in K after a long time the system tends to become stable, i.e. with constant concentration $\hat{x}$.

The main Sacker's result is the following:

- **Theorem** Let $D \subset \mathbf{R}^n$ bounded and $f : \bar{D} \to \mathbf{R}^n$ continuous. Let $int(D)$ the interior of D and $bd(D)$ the boundary of D. Suppose f is injective on $int(D)$, $f(bd(D)) \subset \bar{D}$ and $\mathbf{R}^n \setminus \bar{D}$ has no bounded components. Then $f(\bar{D}) \subset \bar{D}$.

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- The proof for $n = 1$ is a consequence of Darboux theorem, but for $n \geq 2$ the main ingredient is the Brouwer Invariance of Domain Theorem.
- This theorem is applied to the following migration-selection model aiming to obtain an invariant compact K.

An application to a biological migration-selection model

- Let $a_i, b_i > 0$ for $i = 1, 2$ the parameters of a genetic system, such that $\frac{a_1}{b_1} > \frac{a_2}{b_2}$.

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$$\phi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \tau \\ e^{\tau - s} & \text{if } s > \tau \end{cases}$$

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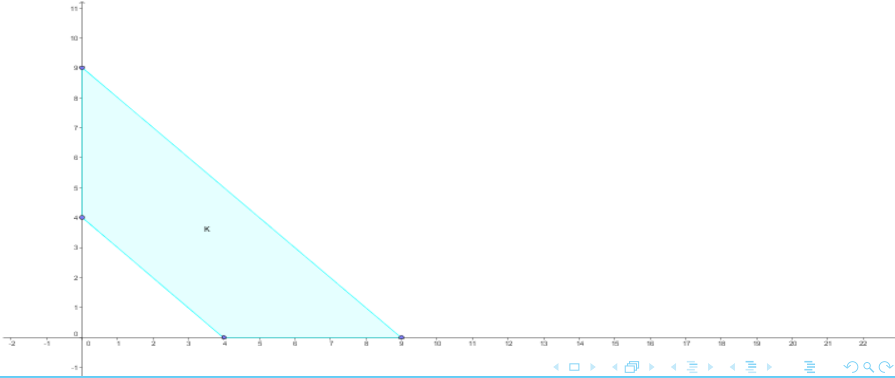
  $$\phi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \tau \\ e^{\tau - s} & \text{if } s > \tau \end{cases}$$
- and $f : \mathbf{R}^2{}_+ \to \mathbf{R}^2{}_+$ $f_i(x_1, x_2) = \frac{(1+a_i)x_i}{1+b_i(x_1+x_2)}\phi(x_1 + x_2)$ the evolution function of the system; recall that this means that

$$x(t+1) = f(x(t)) \qquad (2)$$

Let's consider the region K in the x_1x_2 plane bordered by the axes and by the lines

$$x_1 + x_2 = \frac{a_1}{b_1} \text{ and } x_1 + x_2 = \frac{a_2}{b_2} \tag{3}$$

Compact region K



By direct simple considerations f is injective on $int(K)$ and $f(bd(K)) \subset \bar{K}$. So, by Sacker's theorem we will have $f(\bar{K}) \subset \bar{K}$ as we desired.

Future investigations

I think a good theme for future investigation can be the study of more complicated genetic migration-selection models, for example allowing several types of genes (i.e. $n > 2$) and/or evolution function of greater complexity.

Another direction is to study if the idea we have seen, i. e. to use Brouwer or Leray-Schauder degree to produce invariance of domain type theorems, can be extended in the case of the Conley index - a generalization of the aforementioned invariants.

Acknowledgements

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THANK YOU!