# The Jordan-Brouwer and the Invariance of the domain theorems with modern applications

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# The Jordan-Brouwer theorem and the main steps for proof

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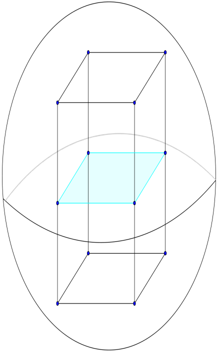
- The fundamental result for the proof is the:

  **Lemma** $\tilde{H}_q(S^n \setminus e^r) = 0 \; \forall q.$
- Let $z$ a $q$ - cycle in $S^n \setminus e^r$. We want that $z$ is a boundary in $S^n \setminus e^r$. The proof of the lemma is done in the following 4 steps:

# First step

Using induction on $r$, we obtain that $z$ is a boundary in $S^n \setminus e^{r-1}(t)$ for any $t$, where $e^{r-1}(t)$ is the image of $I^{r-1} \times \{t\}$.

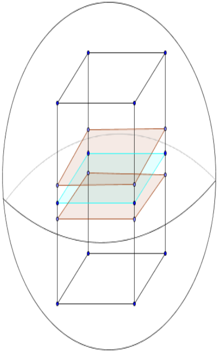
The complement of a fixed floor $t$ in $S^n$



# Second step

Using compacteness arguments and uniform continuity of continuous functions on compact sets, we obtain that for any $t$ there is an $\epsilon_t$ such that $z$ is a boundary in the complement of the image of $I^{r-1} \times (t - \epsilon_t, t + \epsilon_t)$.

The complement of a neighborhood of the $t$ floor in $S^n$



# Third and fourth steps

- **Third step** Using Lebesgue lemma for metric spaces and the previous step, we ca split $[0, 1]$ in $M$ intervals of the form $[j/M, (j+1)/M]$ such that $z$ is a boundary in the complement of the image of $I^{r-1} \times [j/M, (j+1)/M]$ for any $j$.

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- **Fourth step** Using the Mayer-Vietoris sequence and the inductive assumption on $r-1$ we find finally that $z$ is a boundary in $S^n \setminus e^r$.

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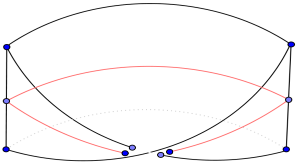
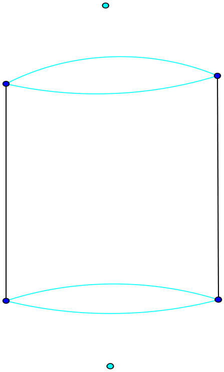
- Are these manifolds diffeomorphic ?
- Of course, the answer is NO! Because one is orientable and the other is not.
- Are these manifolds homeomorphic ?
- The answer is still NO, but the argument is more subtle.

The Moebius strip with median circle and the cylinder with the 2 points used for compactification





In fact if such a homeo would exists, it should send the equator $E$ of the Moebius strip in a simple closed curve $C$ on the cylinder. But compactifying the cylinder and applying Jordan-Brouwer theorem, $C$ will disconnect the cylinder. This is a contradiction with the fact that $E$ does NOT disconnect the Moebius strip.

# The Invariance of the domain theorem

A strong consequence of Jordan-Brouwer theorem is the following:

**Theorem of the invariance of the domain** Let $U$ connected open in $\mathbf{R}^n$ and $f : U \to \mathbf{R}^n$ continuous and injective. Then $f(U)$ is open and $f$ is homeo on its image.

# Three modern applications
# 1. Invariance of Domain Theorem for nonlinear Fredholm maps of index zero between Banach spaces

*After P. Benevieri, M. Furi, M. P. Pera,*
*The invariance of domain for $C^1$ Fredholm maps of index zero.,*
*Recent trends in nonlinear analysis, 35 - 39, Progr. Nonlinear Differential Equations Appl., 40, Birkhauser, Basel, 2000.*

The generalisation presented in the above paper concern Fredholm maps of index 0 between Banach manifolds.

A bounded linear operator between Banach spaces is named Fredholm if both the kernel and the cokernel are finite dimensional.

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- A Fredholm map of index 0 between Banach manifolds is one whose Frechet differential in every point is Fredholm and has index 0.
- **Theorem** Let $M$ and $N$ Banach manifolds and $f : M \to N$ an injective Fredholm map of index 0. Then $f(M)$ is open in $N$.
- The main ingredient for the proof is the Brouwer degree mod-2 for maps between finite dimensional manifolds and its homotopy invariance. A very good reference for this is Milnor's book "Topology from differentiable viewpoint".

# 2. An Invariance of domain theorem for countably condensing vector fields

*After I. S. Kim, The invariance of domain theorem for condensing vector fields, Topological Methods in Nonlinear Analysis 25 (2005), no. 2, 363 - 373.*

Let $E$ a Banach space and $\mathcal{M}$ a collection of subsets of $E$ containing all precompact (i.e. with compact closure) subsets of $E$ and closed under:
closure of convex cover
finite union
finite sum
multiplication by scalars.

- A function $\gamma : \mathcal{M} \to [0, \infty]$ is called a measure of noncompactness on $E$ if:
  $\gamma(\bar{co}A) = \gamma(A)$
  $\gamma(A) = 0$ iff $A$ is precompact
  $\gamma(A \cup B) = max(\gamma(A), \gamma(B))$
  $\gamma(A + B) \leq \gamma(A) + \gamma(B)$
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- In the above setting, for $X$ a subset of $E$ and $k$ a nonnegative real number, a continuous map $F : X \to E$ is called countably $k$-condensing in the strong sense if $F(X) \in \mathcal{M}$ and $\gamma(F(\bar{co}(C))) \leq k\gamma(C)$ for each countable subset $C \subset X$ with $\bar{co}(C) \subset X$ and $C \in \mathcal{M}$.

- **Theorem** Let $X \subset E$ open, $k \in [0,1)$ and $F : X \to E$ a countably k-condensing map in the strong sense, such that $Id - F$ is locally injective. Then $Id - F$ is an open map.($Id$ is the identity of $X$)

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- The main tools in the proof is the Leray-Schauder degree and its homotopy invariance.

## 3. An application to a biological migration-selection model

*After R. J. Sacker, An invariance theorem for mappings, J. Difference Equ. Appl. 18 (2012), no. 1, 163-166.*

Migration-selection models are aimed to describe the change of genetic material across evolution of populations. For example for two types of genes, $G_1$ and $G_2$, their concentrations $x_1$ and $x_2$ are supposed to be modified in time by a low of the following form:

$$x(t+1) = f(x(t)) \tag{1}$$

where $t$ is a particular moment of time, $x = (x_1, x_2)$ is the vector of concentrations and $f$ is a function which depends on the model.

Let $X$ the set of possible concentration vectors. The fundamental question for such a model is the existence of a compact invariant set $K \subset X$ i.e. $f(K) \subset K$.
If such a set exists, the theory of systems of difference equations can be applied to produce a globally attracting fixed point $\hat{x}$. This means a point such that if we start from any concentration in $K$ after a long time the system tends to become stable, i.e. with constant concentration $\hat{x}$.
The main Sacker's result is the following:

- **Theorem** Let $D \subset \mathbf{R}^n$ bounded and $f : \bar{D} \to \mathbf{R}^n$ continuous. Let $int(D)$ the interior of $D$ and $bd(D)$ the boundary of $D$. Suppose $f$ is injective on $int(D)$, $f(bd(D)) \subset \bar{D}$ and $\mathbf{R}^n \setminus \bar{D}$ has no bounded components. Then $f(\bar{D}) \subset \bar{D}$.

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- The proof for $n = 1$ is a consequence of Darboux theorem, but for $n \geq 2$ the main ingredient is the Brouwer Invariance of Domain Theorem.
- This theorem is applied to the following migration-selection model aiming to obtain an invariant compact $K$.

- Let $a_i, b_i > 0$ for $i = 1, 2$ the parameters of a genetic system, such that $\frac{a_1}{b_1} > \frac{a_2}{b_2}$.

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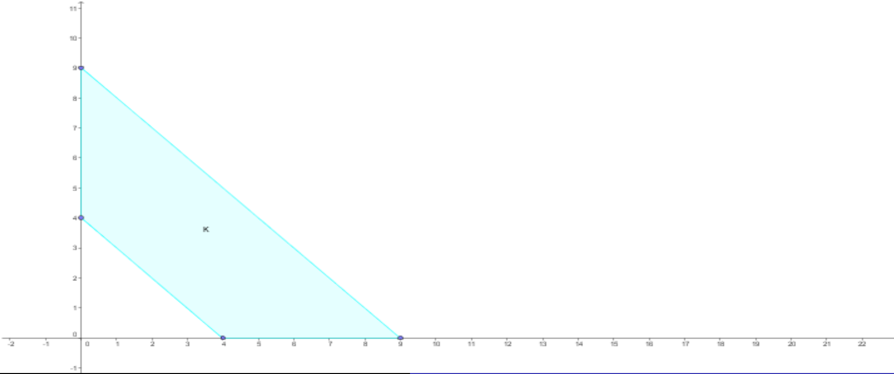
  $$\phi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \tau \\ e^{\tau - s} & \text{if } s > \tau \end{cases}$$
- and $f : \mathbf{R}^2{}_+ \to \mathbf{R}^2{}_+$ $f_i(x_1, x_2) = \frac{(1+a_i)x_i}{1+b_i(x_1+x_2)}\phi(x_1 + x_2)$ the evolution function of the system; recall that this means that

$$x(t + 1) = f(x(t)) \tag{2}$$

Let's consider the region $K$ in the $x_1x_2$ plane bordered by the axes and by the lines

$$x_1 + x_2 = \frac{a_1}{b_1} \textit{ and } x_1 + x_2 = \frac{a_2}{b_2} \tag{3}$$

## Compact region K



By direct simple considerations $f$ is injective on $int(K)$ and $f(bd(K)) \subset \bar{K}$. So, by Sacker's theorem we will have $f(\bar{K}) \subset \bar{K}$ as we desired.

# Future investigations

I think a good theme for future investigation can be the study of more complicated genetic migration-selection models, for example allowing several types of genes (i.e. $n > 2$) and/or evolution function of greater complexity.

Another direction is to study if the idea we have seen, i. e. to use Brouwer or Leray-Schauder degree to produce invariance of domain type theorems, can be extended in the case of the Conley index - a generalization of the aforementioned invariants.

# Acknowledgements

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THANK YOU!